Self-Alignment Techniques for Strapdown Inertial Navigation Systems with Aircraft Application

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One of the more critical problem areas in the application of strapdown inertial techniques to the navigation of commercial aircraft is that of initial alignment. A two-stage self-alignment scheme that appears promising in this regard is explored. The first or "coarse" alignment stage utiizes the measurement of the gravity and earth rotation vectors to directly compute the transformation matrix relating the body frame to a reference frame. A linearized error analysis is presented. The second "fine" alignment stage corrects the initial transformation estimate by supplying a base motion compensated angular velocity signal to the transformation computer. This correction signal is computed by using estimates of the error angles between a known reference frame and the corresponding computed frame. Kalman filtering techniques are used to implement this technique and an error analysis is presented.

Nomenclature

$\mathbf{C}$ = coordinate transformation matrix
$\mathbf{f}$ = specific force vector
$\delta\mathbf{f}$ = uncertainties in the measured specific force caused by the accelerometer uncertainties and the disturbance accelerations
$(u)\mathbf{f}$ = accelerometer bias
$\mathbf{g}$ = gravity field vector
L = geographic latitude
$\mathbf{M}$ = constant coefficient matrix
$\mathbf{m}$ = measurement vector
$\delta\mathbf{m}$ = error in measurement vector
$\delta\mathbf{m}_C$ = random measurement bias
$\mathbf{v}$ = uncorrelated noise vector
$\epsilon_N,\epsilon_E,\epsilon_D$ = misalignment angles between the n and n' frames about the positive north, east and down axes, respectively
$\epsilon_x,\epsilon_y,\epsilon_z$ = misalignment angles between the b and b' frames about the positive x, y, and z body axes, respectively
$\boldsymbol{\varepsilon}$ = misalignment angle vector
$\delta\boldsymbol{\varepsilon}$ = uncertainties in the measured error angles
$\boldsymbol{\varepsilon}(t_i^-)$ = value of misalignment angle before update
$\boldsymbol{\varepsilon}(t_i^+)$ = value of misalignment angle after update
$\mathbf{E}$ = skew symmetric form of misalignment angles
$\boldsymbol{\omega}_{ie}$ = Earth angular velocity vector
$(u)\boldsymbol{\omega}$ = gyro drift
$\delta\boldsymbol{\omega}$ = uncertainties in the measured Earth angular velocity
Ω = skew symmetric angular velocity matrix
$\delta(\ )$ = Dirac delta function
$\mathbf{R}$ = covariance matrix of $\mathbf{v}$

Sub/superscripts

b = body coordinate frame
b' = computed body coordinate frame
d = disturbance quantity
i = inertial coordinate frame
n = geographic coordinate frame
n' = computed geographic coordinate frame
N,E,D = components of vectors resolved in geographic coordinate frame
x,y,z = components of vectors resolved in body coordinate frame
T = transpose
$*$ = orthogonalized quantity
$\frown{(\ )}$ = arc length measure
$\bar{(\ )}$ = mathematical expectation
$\hat{(\ )}$ = estimate value

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Introduction

THE current interest in analytical or strapdown inertial navigation systems for aircraft application is directly attributable to the revolution in computer technology that now allows solution of the navigation equations with a machine whose physical characteristics are compatible with aircraft requirements. The concepts involved in the design of such a system have been thoroughly explored in the literature[1,2] whereas the error analysis and detailed application is a subject of current research.[3–5] Analytic systems offer the posibility of higher reliability than present gimballed systems since redundancy can be provided at the component level rather than at the subsystem level. Thus, the analytic inertial navigation system, augmented by appropriate navigation aids to bound the inherent long term error, is receiving serious consideration for application in the advanced supersonic transport.

One of the more critical problem areas in this application is that of initial alignment within the environment and time constraints imposed by commercial aircraft operation. That is, the system must be aligned within the necessary tolerances in the short period of time necessary for commercial success of the aircraft in the face of deleterious motions of the aircraft caused by wind gusts, the loading of passengers and cargo, fuel ingestion, etc. This paper will explore a two stage alignment technique that appears to be promising in this regard. The problem of alignment in a strapdown inertial guidance system is basically that of determining the transformation matrix, that relates vectors in the instrumented body coordinate frame b frame to the same vectors expressed in inertial coordinates i frame or equivalently, in some other computation frame. Two methods for self-contained alignment will be considered here. Both of these use the fact that two vectors uniquely determine the transformation matrix between two coordinate frames, if they are known in both frames and are not colinear. The two procedures are as follows.

Analytic Alignment

The transformation matrix is computed directly using the knowledge of $\mathbf{g}$ and $\boldsymbol{\omega}_{ie}$, i.e., the gravity and earth rotation vectors, in the two frames. These vectors are known in the inertial frame to the accuracy of the time measuring device used. The body frame coordinates are measured by the accelerometers and gyros and therefore contain instrument uncertainties.

Corrective Alignment

If it may be assumed that some estimate of the transformation matrix is available, then this initial matrix can be corrected by feeding an appropriate rate signal to the transformation computer. This correction signal can be computed by using estimates of the error angles between a known reference frame and the corresponding computed frame.

For either alignment scheme, the instrumented frame is taken to be stationary with respect to the Earth, except for disturbances that occur due to wind gusts or loading and unloading of the vehicle. However, no data is available at this time on the motion of such or a similar aircraft due to wind gusts and other disturbances on the ground. The two methods will now be treated in detail.

Analytic Alignment

Description

It is convenient to split the transformation between the b and i frames into two parts by using the local vertical or geographic coordinate system n frame, as an intermediate frame. The transformation of the specific force vector $\mathbf{f}$ can then be visualized as shown in Fig. 1, where the vector superscript refers to the coordinate system in which the vectors are resolved. The transformation $\mathbf{C}_n{}^i$, i.e., from n frame to i frame, is known as a function of time for any given latitude and longitude. The transformation $\mathbf{C}_b{}^n$ remains to be determined. The $\mathbf{C}_b{}^n$ matrix can be found by the estimation of two vectors; namely, the earth rate vector $\boldsymbol{\omega}_{ie}$ and the gravity vector $\mathbf{g}$ in the two frames of interest. For the body frame these vectors are obtained by the gyros and the accelerometers, respectively, and in the navigation frame they are known and constant, which makes this frame a convenient reference. The gravity and angular rate vectors transform according to the following expressions:

$$\mathbf{g}^b = \mathbf{C}_n{}^b\mathbf{g}^n, \ \boldsymbol{\omega}_{ie}{}^b = \mathbf{C}_n{}^b\boldsymbol{\omega}_{ie}{}^n$$

If $\mathbf{v}$ is defined as $\mathbf{v} = \mathbf{g} \times \omega_{ie}$, we also have

$$\mathbf{v}^b = \mathbf{C}^b{}_n\mathbf{v}^n$$

Since $\mathbf{C}_b{}^n = (\mathbf{C}_n{}^b)^{-1} = (\mathbf{C}_n{}^b)^T$ these three vector relations can be written

$$\mathbf{C}_b{}^n = \begin{bmatrix} (\mathbf{g}^n)^T \\ \cdots \\ (\boldsymbol{\omega}_{ie}{}^n)^T \\ \cdots \\ (\mathbf{v}^n)^T \end{bmatrix}^{-1} \begin{bmatrix} (\mathbf{g}^b)^T \\ \cdots \\ (\boldsymbol{\omega}_{ie}{}^b)^T \\ \cdots \\ (\mathbf{v}^b)^T \end{bmatrix} \quad (1)$$

Thus, the alignment matrix is uniquely defined provided that the inverse indicated previously exists. This inverse exists if no one row of the matrix is a linear combination of the remaining rows. This condition is always satisfied if the two vectors $\mathbf{g}$ and $\boldsymbol{\omega}_{ie}$ are not collinear. These vectors coincide only at the Earth's poles, where the analytic alignment procedure is useless. For the general case, the inverse in Eq. (1) is given by

$$\begin{bmatrix} (\mathbf{g}^n)^T \\ (\boldsymbol{\omega}_{ie}{}^n)^T \\ (\mathbf{v}^n)^T \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & g \\ \omega_{ie}\cos L & 0 & -\omega_{ie}\sin L \\ 0 & g\omega_{ie}\cos L & 0 \end{bmatrix}^{-1} = \begin{bmatrix} (1)/g)\tan L & (1/\omega_{ie})\sec L & 0 \\ 0 & 0 & (1/g\omega_{ie})\sec L \\ 1/g & 0 & 0 \end{bmatrix} \quad (2)$$

Kasper[6] shows that for fixed base alignment, the analytic scheme compares favorably with the existing optical alignment methods.

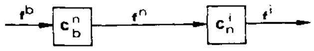

Fig. 1 Specific force transformation.

In the present case, however, its performance deteriorates because of the angular disturbance vibrations and accelerations. The effect is two-fold; first, the disturbances corrupt the measurements of $\mathbf{g}^b$ and $\boldsymbol{\omega}_{ie}{}^b$ since the measured quantities are

$$\mathbf{f} = \mathbf{g} + \mathbf{f}_d$$

$$\boldsymbol{\omega}_{ib} = \boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_d$$

where d indicates the disturbance quantities; secondly, $\mathbf{g}^b$ and $\boldsymbol{\omega}_{ie}{}^b$ become functions of time to some extent. This can be seen from the fact that since $\dot{\boldsymbol{\omega}}_{ie}{}^n = 0$,

$$\dot{\boldsymbol{\omega}}_{ie}{}^b = -\boldsymbol{\Omega}_{nb}{}^b\boldsymbol{\omega}_{ie}{}^b$$

where the elements of the skew symmetric matrix $\boldsymbol{\Omega}_{nb}{}^b$ are given by the components of $\boldsymbol{\omega}_d$. It is, therefore, necessary to introduce some filtering in order to reduce the effects of these vibrations. A simple low-pass filter could be used to obtain the average values of the measured quantities. This would tend to give the average alignment matrix. It is clear, however, that the instantaneous position of the body frame can vary considerably from its average position, depending upon the motion of the aircraft. As a result, a large initial misalignment could exist when the system is switched to the navigation mode of operation if only an average alignment were achieved. If the statistics of the aircraft vibrations were available, a more elaborate optimal filtering scheme could be constructed. However, it could prove difficult to separate the perturbations of $\boldsymbol{\omega}_{ie}{}^b$ and $\mathbf{g}^b$ from the disturbances $\boldsymbol{\omega}_d$ and $\mathbf{f}_d$ by linear filtering, since it is very likely that these components contain the same frequencies. In addition some time lag would be introduced by the filter. The analytic alignment method is therefore mainly useful as an average alignment, which is a rapid way of obtaining an initial estimate of the transformation matrix.

Error Analysis

An error analysis for this alignment scheme, that takes into account the effect of instrument uncertainties and base motion is not readily amenable to analytic methods. The analysis that follows is intended to indicate an approach that will result in equations that are best solved on a digital computer. The equation for $\mathbf{C}_b{}^n$ can be written in the form

$$\mathbf{C}_b{}^n = \mathbf{MQ} \quad (3)$$

where

$$\mathbf{M} = \begin{bmatrix} (\mathbf{g}^n)^T \\ (\boldsymbol{\omega}_{ie}{}^n)^T \\ (\mathbf{v}^n)^T \end{bmatrix}^{-1} \text{ and } \mathbf{Q} = \begin{bmatrix} (\mathbf{f}^b)^T \\ (\boldsymbol{\omega}_{ib}{}^b)^T \\ (\mathbf{v}^b)^T \end{bmatrix}$$

The elements of $\mathbf{M}$ are constant at any latitude and are given by Eq. (2), but $\mathbf{Q}$ contains measurement uncertainties. The above equation can therefore be written as

$$\mathbf{C}_b{}^{n\prime} = \mathbf{M}(\mathbf{Q} + \delta\mathbf{Q})$$

where

$$\delta\mathbf{Q} = \begin{bmatrix} \delta\mathbf{f}^T \\ \delta\boldsymbol{\omega}^T \\ \delta\mathbf{v}^T \end{bmatrix} = \begin{bmatrix} \delta f_x & \delta f_y & \delta f_z \\ \delta\omega_x & \delta\omega_y & \delta\omega_z \\ \delta\nu_x & \delta\nu_y & \delta\nu_z \end{bmatrix} \quad (4)$$

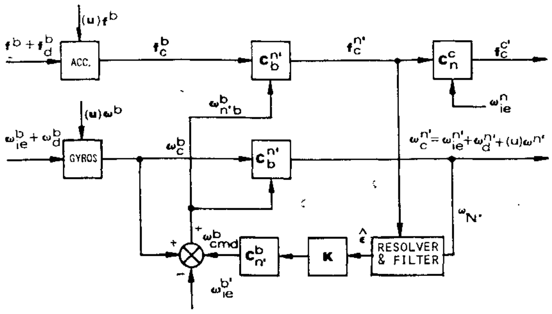

Fig. 2 Self-corrective alignment scheme.

Thus, $\mathbf{C}^{n'}{}_b = \mathbf{C}^{n'}{}_n \mathbf{C}_b{}^n = (\mathbf{I} + \mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b)\mathbf{C}_b{}^n$, which shows that

$$\mathbf{C}^{n'}{}_n = (\mathbf{I} + \mathbf{M}\delta\mathbf{Q}\ \mathbf{C}_n{}^b) \tag{5}$$

If the lengths of the measured vectors $\mathbf{g}^b$, $\boldsymbol{\omega}_{ie}{}^b$, and $\boldsymbol{\nu}^b$ and the angles between them are required to be constant, then the matrix $\mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b$ must necessarily be of the skew symmetrical form

$$\mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b = \begin{bmatrix} 0 & \epsilon_D & -\epsilon_E \\ -\epsilon_D & 0 & \epsilon_N \\ \epsilon_E & -\epsilon_N & 0 \end{bmatrix} = -E_n$$

When Eqs. (2) and (4) are substituted into Eq. (5) the resulting $\mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b$ matrix is of course, not in the desired skew symmetric form, since

$$\mathbf{M}\delta\mathbf{Q} = \begin{bmatrix} (\delta f_x/g)\tan L + (\delta\omega_x/\omega_{ie})\sec L & (\delta f_y/g)\tan L + (\delta\omega_y/\omega_{ie})\sec L & (\delta f_z/g)\tan L + (\delta\omega_z/\omega_{ie})\sec L \\ (\delta\nu_x/g\omega_{ie})\sec L & (\delta\nu_y/g\omega_{ie})\sec L & (\delta\nu_z/g\omega_{ie})\sec L \\ \delta f_x/g & \delta f_y/g & \delta f_z/g \end{bmatrix} \tag{6}$$

where

$$\delta\nu_x = g_y\delta\omega_z + \omega_z\delta f_y - g_z\delta\omega_y - \omega_y\delta f_z$$

$$\delta\nu_y = g_z\delta\omega_x + \omega_x\delta f_z - g_x\delta\omega_z - \omega_z\delta f_x$$

$$\delta\nu_z = g_x\delta\omega_y + \omega_y\delta f_x - g_y\delta\omega_x - \omega_x\delta f_y$$

Moreover, it is difficult to directly apply the constraints that the lengths of the measured vectors and the angles between them be constant. This problem is resolved in a practical manner by requiring that the computed transformation $\mathbf{C}_b{}^{n'}$ be orthogonal. This is accomplished by setting[4]

$$\mathbf{C}^* = \mathbf{C}^{n'}{}_b[(\mathbf{C}^{n'}{}_b)^T\ (\mathbf{C}^{n'}{}_b)]^{1/2} \tag{7}$$

where $\mathbf{C}^* \sim$ optimal orthogonal approximation to $\mathbf{C}^{n'}{}_b$ in the sense that trace $(\mathbf{C}^* - \mathbf{C}^{n'}{}_b)^T\ (\mathbf{C}^* - \mathbf{C}^{n'}{}_b)$ is minimized. Since from Eq. (5),

$$\mathbf{C}^{n'}{}_b = (\mathbf{I} + \mathbf{M}\delta\mathbf{Q}\ \mathbf{C}_n{}^b)\ \mathbf{C}_b{}^n$$

then

$$\mathbf{C}^* = (\mathbf{I} + \mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b)\ \mathbf{C}_b{}^n(\mathbf{I} + \delta\mathbf{Q}^T\mathbf{M}^T\mathbf{C}_b{}^n + \mathbf{C}_n{}^b\ \mathbf{M}\delta\mathbf{Q})^{-1/2}$$

where products of error quantities have been neglected. If the square root is expanded in series, there results

$$\mathbf{C}^* = [\mathbf{I} + \tfrac{1}{2}(\mathbf{M}\delta\mathbf{Q}\mathbf{C}_n{}^b - \mathbf{C}_b{}^n\ \delta\mathbf{Q}^T\mathbf{M}^T)]\mathbf{C}_b{}^n$$

which is, by inspection, in the desired skew symmetric form. Unfortunately, for arbitrary orientation of the b frame relative to the n frame, the analytic expressions for the misalignment angles as a function of the instrum certainties are quite messy and not readily amenable to physical interpretation.

A particular simple result emerges if the body frame is taken to be aligned with the local geographic frame $\mathbf{C}_b{}^n = \mathbf{I}$. In this case, it was found that

$$\mathbf{C}^* = \left\{\mathbf{I} - \begin{bmatrix} 0 & -\epsilon_D{}^* & \epsilon_E{}^* \\ \epsilon_D & 0 & -\epsilon_N{}^* \\ -\epsilon_E{}^* & \epsilon_N{}^* & 0 \end{bmatrix}\right\}\mathbf{C}_b{}^n \tag{9}$$

where $\epsilon_j{}^*$, $j = N,E,D$ are the misalignment angles which result after the orthogonalizing procedure and are given by

$$\epsilon_N{}^* = -\delta f_y/g$$

$$\epsilon_E{}^* = \tfrac{1}{2}\ [(\delta f_x/g) - (\delta f_z/g)\tan L - (\delta\omega_z/\omega_{ie})\sec L]$$

$$\epsilon_D{}^* = (\delta f_y/g)\tan L + (\delta\omega_y/\omega_{ie})\sec L$$

Thus, it is seen that one can expect to see a north level error of about 3.4 min/mg east accelerometer uncertainty. The east level error due to north accelerometer uncertainty is 1.7 min/mg; to azimuth accelerometer uncertainty, 1.7 min tan$L$/mg; and to azimuth gyro uncertainty 1.7 min sec$L$/meru. The azimuth error angle due to the east accelerometer uncertainty is given by -3.4 min tan$L$/mg, and due to east gyro uncertainty is given by -3.4 min sec$L$/meru.

If these results are compared with those that would be obtained with an acceleration coupled platform gyrocompass,[9] it is seen that the north level error $\epsilon_N{}^*$ is identical to that which would be obtained using a platform gyrocompass. The east level error $\epsilon_E{}^*$ contains additional terms because of the orthogonalizing procedure that are functions of the vertical accelerometer and azimuth gyro uncertainties. Finally, the azimuth error $\epsilon_D{}^*$ is seen to be identical for the two gyrocompassing systems.

It is also interesting to compare the results of Eq. (9) with those which would be obtained if the instrument package could be rotated such that two accelerometers were nulled and the angular velocity about indicated east were zero. In this situation, the conditions for equilibrium are found to be

$$\delta f_y + g\epsilon_N = 0,\ \delta f_x - g\epsilon_E = 0$$

$$\delta\omega_y - \omega_{ie}\cos L\ \epsilon_D - \omega_{ie}\sin L\epsilon_N = 0$$

These results are seen to be the same as those obtained for the acceleration coupled gyro compass of Ref. 9.

Corrective Alignment

Description

This procedure can be mechanized in the following manner, using the local vertical navigation frame as a reference frame (see Fig. 2). As before the transformation consists of two steps; i.e., from the instrumented body frame b to the navigation frame n and then to the inertial frame used for computations.

It is assumed here that an initial estimate of the transformation matrix is available that corresponds to a small angle misalignment of the computed and actual reference frame. The method, basically, consists of detecting the error angles between these two frames via the processed accelerometer and gyro signals and generating a signal to the transformation computer in order to reduce these angles as close to zero as possible. At the same time, compensation must be provided for the disturbance angular vibrations similar to the base isolation of a gimballed platform system. This is accomplished by feeding the measurements of $\boldsymbol{\omega}_d{}^b$ to the transformation computer as shown in Fig. 2. The transformation matrix $\dot{\mathbf{C}}^{n'}{}_b$ is updated using the relation

$$\dot{\mathbf{C}}^{n'}{}_b = \mathbf{C}^{n'}{}_b\ \boldsymbol{\Omega}_{n'b}{}^b \tag{10}$$

where $\boldsymbol{\Omega}_{n'b}{}^b$ is a skew symmetric matrix of the angular velocity $\boldsymbol{\omega}_{n'b}{}^b$, that is fed to the transformation computer.

This angular velocity signal would ideally be

$$\boldsymbol{\omega}_{n'b}{}^{b} = \boldsymbol{\omega}_{cmd}{}^{b} + \boldsymbol{\omega}_{d}{}^{b}$$

where $\boldsymbol{\omega}_{cmd}$ is the computed correction signal and $\boldsymbol{\omega}_d{}^b$ compensates for the vibrations of the instrumented body frame. In Fig. 2 $\boldsymbol{\omega}_d{}^b$ is obtained by subtracting $\boldsymbol{\omega}^{b'}{}_{ie}$ from the total angular velocity, but since $\boldsymbol{\omega}^{b'}{}_{ie}$ is not equal to $\boldsymbol{\omega}_{ie}{}^b$ and the signal, in addition, contains the gyro drift $(u)\boldsymbol{\omega}^b$, $\boldsymbol{\omega}_{n'b}{}^b$ becomes

$$\boldsymbol{\omega}_{n'b}{}^{b} = \boldsymbol{\omega}_{cmd}{}^{b} + \boldsymbol{\omega}_{d}{}^{b} - E^b\boldsymbol{\omega}_{ie}{}^b + (u)\boldsymbol{\omega}^b \tag{12}$$

Here, $\mathbf{E}^b$ is defined as the skew symmetric matrix of the misalignment angles between the actual and computed body frames, b and b'

$$\mathbf{E}^b = \begin{bmatrix} 0 & -\epsilon_z & \epsilon_y \\ \epsilon_z & 0 & -\epsilon_x \\ -\epsilon_y & \epsilon_x & 0 \end{bmatrix}$$

Error Analysis

The error angle equations can now be derived; substituting the skew symmetric form of Eq. (12) into Eq. (10) yields

$$\dot{\mathbf{C}}^{n'}{}_{b} = \mathbf{C}^{n'}{}_b\boldsymbol{\Omega}_{cmd}{}^b + \mathbf{C}^{n'}{}_b\boldsymbol{\Omega}_d{}^b - \mathbf{C}^{n'}{}_b\delta\boldsymbol{\Omega}_{ie}{}^b + \mathbf{C}^{n'}{}_b(u)\boldsymbol{\Omega}^b \tag{13}$$

where $\delta\boldsymbol{\Omega}_{ie}{}^b$ is the skew symmetric form of $\mathbf{E}^b\boldsymbol{\omega}_{ie}{}^b$. Noting that

$$\dot{\mathbf{C}}^{n'}{}_{b} = \mathbf{C}^{n'}{}_n\dot{\mathbf{C}}_b{}^n + \dot{\mathbf{C}}^{n'}{}_n\mathbf{C}_b{}^n$$

and

$$\dot{\mathbf{C}}_b{}^n = \mathbf{C}_b{}^n\boldsymbol{\Omega}_d{}^b \tag{14}$$

since the b frame rotates with an angular velocity of $\boldsymbol{\omega}_d$ with respect to the n frame, Eq. (13) then becomes

$$\dot{\mathbf{C}}^{n'}{}_n\mathbf{C}_b{}^n = \mathbf{C}^{n'}{}_b\boldsymbol{\Omega}_{cmd}{}^b - \mathbf{C}^{n'}{}_b\delta\boldsymbol{\Omega}_{ie}{}^b + \mathbf{C}^{n'}{}_b(u)\boldsymbol{\Omega}^b \tag{15}$$

Using the fact that $\mathbf{C}^{n'}{}_n = \mathbf{I} - \mathbf{E}^n$ and $\mathbf{E}^n = \mathbf{C}_b{}^n\mathbf{E}^b\mathbf{C}_n{}^b$, Eq. (15) reduces to

$$\dot{\mathbf{E}}^b = -\boldsymbol{\Omega}_{cmd}{}^b - (u)\boldsymbol{\Omega}^b + \delta\boldsymbol{\Omega}_{ie}{}^b \tag{16}$$

or equivalently for $\mathbf{E}^n$

$$\dot{\mathbf{E}}^n = \boldsymbol{\Omega}_{cmd}{}^n - (u)\boldsymbol{\Omega}^n + \delta\boldsymbol{\Omega}_{ie}{}^n \tag{17}$$

where higher-order terms have been neglected.

In order to drive $\mathbf{E}^n$ to zero, $\boldsymbol{\omega}_{cmd}{}^{n'}$ can be chosen to be a linear function of the measured estimate of $\mathbf{E}^n$. The vector form of Eq. (17) then becomes

$$\boldsymbol{\varepsilon}^n + K\hat{\boldsymbol{\varepsilon}}^n = -(u)\boldsymbol{\omega}^n - \boldsymbol{\Omega}_{ie}{}^n\boldsymbol{\varepsilon}^n \tag{18}$$

where $\mathbf{K}$ remains to be specified, and $\hat{\boldsymbol{\varepsilon}}$ indicates an estimate of $\boldsymbol{\varepsilon}$; ($\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} + \delta\boldsymbol{\varepsilon}$). It is noted that the three scalar equations are coupled through the term $\boldsymbol{\Omega}_{ie}{}^n\boldsymbol{\varepsilon}^n$.

The error angle vector, $\boldsymbol{\varepsilon}^n$ must now be measured. A direct indication of the three components can be obtained from the computed horizontal components of $\mathbf{g}$ and the computed east component of $\boldsymbol{\omega}_{ie}$, which are approximately proportional to ϵ_N, ϵ_E, and ϵ_D. Specifically, since

$$\mathbf{f}^{n'}{}_c = \mathbf{C}^{n'}{}_b\mathbf{f}_c{}^b = (\mathbf{I} - \mathbf{E}^n)\mathbf{f}_c{}^n = (\mathbf{I} - \mathbf{E}^n)[\mathbf{f}_d + \mathbf{f}_n{}^n + (u)\mathbf{f}^n]$$

and $\mathbf{f}^n = (0,0,g)$, then

$$\mathbf{f}_N{}' = -g\epsilon_E + \delta f_N \tag{19a}$$

$$f_E{}' = g\epsilon_N + \delta f_E \tag{19b}$$

where δf_N and δf_E represent the uncertainties in the computed north and east specific force components that are caused by the accelerometer uncertainties and the disturbance accelerations. In a similar fashion, the computed east component of Earth rate is given by

$$\omega_E{}' = -\omega_{ie}\cos L(\epsilon_D + \epsilon_N \tan L) + \delta\omega_E \tag{19c}$$

This arrangement for the extraction of $\boldsymbol{\varepsilon}$ is shown in Fig. 2. Alternatively, the vector products of the measured and actual vectors, $\mathbf{g}^n \times \mathbf{f}^{n'}$ and $\boldsymbol{\omega}_{ie}{}^n \times \boldsymbol{\omega}^{n'}{}_{ie}$, could be used to indicate the misalignment.

It is now necessary to determine the form of the $\mathbf{K}$ matrix in Eq. (18). Assuming for this purpose that no uncertainties exist, Eq. (18) becomes

$$\dot{\boldsymbol{\varepsilon}}^n = -\boldsymbol{\Omega}_{ie}{}^n\boldsymbol{\varepsilon}^n - \mathbf{K}\boldsymbol{\varepsilon}^n \tag{18a}$$

which can be identified with the general form

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{u} \tag{20}$$

where $\mathbf{x} = \boldsymbol{\varepsilon}^n$, $\mathbf{F} = -\boldsymbol{\Omega}_{ie}{}^n$, $\mathbf{u} = -\mathbf{K}\boldsymbol{\varepsilon}^n$. One way of determining $\mathbf{K}$ is to define a cost function of the form

$$J = \tfrac{1}{2}\int_{t_0}^{\infty} (\mathbf{x}^T\mathbf{U}\mathbf{x} + \mathbf{u}^T\mathbf{W}\mathbf{u})dt \tag{21}$$

where t_0 is the starting time of the process, and $\mathbf{U}$ and $\mathbf{W}$ are positive definite weighting matrices which will be chosen to to be constant here. The task of $\mathbf{U}$ and $\mathbf{W}$ is to insure that $\mathbf{x}$ and $\mathbf{u}$ remain within acceptable levels. Bryson[7] suggests that a reasonable choice for $\mathbf{U}$ and $\mathbf{W}$ may be

$$\mathbf{U} = \begin{bmatrix} \frac{1}{x_{1m}{}^2} & 0 & 0 \\ 0 & \frac{1}{x_{2m}{}^2} & 0 \\ 0 & 0 & \frac{1}{x_{3m}{}^2} \end{bmatrix} \text{ and } \mathbf{W} = \begin{bmatrix} \frac{1}{u_{1m}{}^2} & 0 & 0 \\ 0 & \frac{1}{u_{2m}{}^2} & 0 \\ 0 & 0 & \frac{1}{u_{3m}{}^2} \end{bmatrix}$$

x_{im} and u_{im} are the maximum acceptable levels of the state and control.

It can be shown[7] that in order to minimize the performance index, $\mathbf{K}$ should have the value

$$\mathbf{K} = \mathbf{W}^{-1}\mathbf{S} \tag{22}$$

where $\mathbf{S}$ is the solution of the steady-state Ricatti equation

$$\mathbf{S}\mathbf{F} + \mathbf{F}^T\mathbf{S} - \mathbf{S}\mathbf{W}^{-1}\mathbf{S} + \mathbf{U} = \mathbf{0} \tag{23}$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & -\omega_{ie}\sin L & 0 \\ \omega_{ie}\sin L & 0 & \omega_{ie}\cos L \\ 0 & -\omega_{ie}\cos L & 0 \end{bmatrix}$$

Equation (23) must in general be solved by numerical method. Another way of determining $\mathbf{K}$ is to require that Eq. (18) become uncoupled. This can be done approximately sinec $\boldsymbol{\omega}_{ie}{}^n$ is constant for a given latitude, so by choosing the off-diagonal terms of $\mathbf{K}$ equal to the corresponding terms of the skew symmetric matrix $-\boldsymbol{\Omega}^n{}_{ie}$, Eq. (18) becomes

$$\dot{\boldsymbol{\varepsilon}}^n + \mathbf{K}_d\boldsymbol{\varepsilon}^n = -(u)\boldsymbol{\omega}^n - \mathbf{K}\delta\boldsymbol{\varepsilon}^n \tag{24}$$

where $\mathbf{K}_d = \mathbf{K} + \boldsymbol{\Omega}^n{}_{ie}$, a diagonal matrix.

The effect of measurement uncertainties and gyro drift on $\boldsymbol{\varepsilon}$ is then, neglecting the weak effect of the off-diagonal terms of $\mathbf{K}\delta\boldsymbol{\varepsilon}$:

$$\dot{\boldsymbol{\varepsilon}}^n + \mathbf{K}_d\boldsymbol{\varepsilon}^n = -(u)\boldsymbol{\omega}^n - \mathbf{K}_d\delta\boldsymbol{\varepsilon}^n \tag{24a}$$

If the statistics of the disturbances $\boldsymbol{\omega}_d$ and $\mathbf{f}_d$ were known, it would be possible to construct a filter that separates the signal from the noise in some optimum way. A low-pass filter that approximately averages the input, may attenuate $\boldsymbol{\omega}_d$ sufficiently, but since $(u)\omega_y$, $(u)f_x$, and $(u)f_y$ are expected to be very nearly constant during the alignment there will

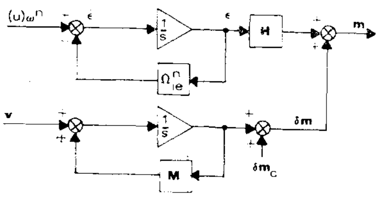

Fig. 3 System error model.

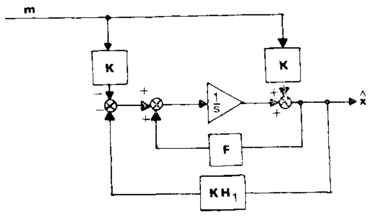

Fig. 4 Optimum linear filter.

be a steady-state alignment error due to gyro drift and accelerometer bias.

Filtering

The main problem in the alignment process is to obtain sufficiently good estimates of the error angles. After such an estimate has been made, the transformation matrix can be corrected almost instantaneously (assuming small angles). This suggests that the updating could be performed discretely rather than continuously. The in between intervals would then be used for the estimation process. The correction signals would consist of the angular rotations about each axis needed to align the two frames in question. The error equation in this case is

$$\dot{\boldsymbol{\varepsilon}}^n = -\boldsymbol{\Omega}_{ie}{}^n \boldsymbol{\epsilon}^n - (u)\boldsymbol{\omega}^n \tag{25}$$

between updates and

$$\boldsymbol{\varepsilon}^n(t_i^+) = \boldsymbol{\varepsilon}^n(t_i^-) - \hat{\boldsymbol{\varepsilon}}^n(t_i^-) \tag{26}$$

at the update time t_i. The length of the interval depends on how much time is needed to get a satisfactory estimate of the error angles. If, in the estimation of ϵ_D for example, $\omega_{d_y} = A \sin\omega t$, the average $\bar{\omega}_{d_y}$ is zero, but the computed average would be

$$\hat{\omega}_{d_y} = \frac{1}{T}\int_0^T A \sin\omega t dt = \frac{A}{2\pi}\frac{\tau}{T}[1 - \cos\omega T]$$

In this case, T would therefore have to be long enough compared to $\tau = 2\pi/\omega$, to bring this average reasonably close to zero. The filtering time thus more or less determines the alignment time. Note that even without any disturbances some filtering is necessary because of instrument noise.

Instead of a simple averaging procedure, a more complex optimum filtering technique could be used. If it may be assumed that the disturbances and instrument uncertainties are Gaussian, a Kalman type filter can be constructed for the estimation of the error angles. As an example, the instrument uncertainties are assumed to be a normally distributed random constant while the external disturbances are taken to be exponentially correlated and can be represented by passing uncorrelated noise through a first-order system. The modelling of these correlated signals effectively increases the order of the system such that the instrument uncertainties and external disturbances are now system state variables whose estimates are obtained in addition to the estimates of $\boldsymbol{\varepsilon}^n$. This expanded system is shown in Fig. 3, where

$$\mathbf{m} = \{f_E', f_N', \omega_E'\}$$

$$\mathbf{H} = \begin{bmatrix} g & 0 & 0 \\ 0 & -g & 0 \\ -\omega_{ie}\sin L & 0 & -\omega_{ie}\cos L \end{bmatrix}$$

and

$$\overline{\mathbf{v}(t)\mathbf{v}^T(\tau)} = \mathbf{R}\delta(t - \tau)$$

Since the measurement uncertainties $\boldsymbol{\delta}\mathbf{m}$ are correlated the application of Kalman filtering is somewhat modified.[10] It can be shown that the appropriate form of the filter in this case is represented by Fig. 4, where

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\boldsymbol{\varepsilon}}^n \\ \widehat{(u)\boldsymbol{\omega}^n} \\ \widehat{\delta \mathbf{m}} \end{bmatrix}; \mathbf{F} = \begin{bmatrix} -\boldsymbol{\Omega}_{ie}{}^n & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{bmatrix}$$

$$\mathbf{H}_1 = [-\mathbf{H}\,\boldsymbol{\Omega}_{ie}{}^n \,\vert\, \mathbf{H} \,\vert\, \mathbf{M}]$$

and $\mathbf{K}$ and $\dot{\mathbf{K}}$ are determined by the equations

$$\mathbf{K} = (\mathbf{P}\mathbf{H}_1{}^T + \mathbf{B})\mathbf{R}^{-1}$$

$$\dot{\mathbf{K}} = \dot{\mathbf{P}}\mathbf{H}_1{}^T\mathbf{R}^{-1}$$

where

$$\mathbf{B} = \mathbf{G}\mathbf{R} \qquad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

$$\mathbf{P} = \overline{(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T}$$

and is found by solving the differential equation

$$\dot{\mathbf{P}} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{G}\mathbf{R}\mathbf{G}^T - \mathbf{K}\mathbf{R}\mathbf{K}^T$$

when $P(t_0) = P_0$ is a known initial condition matrix. The estimated values $\hat{\mathbf{x}}$ would then be used to periodically up date the transformation matrix and with estimates of instrument uncertainties available compensation for these error sources could be provided. Obviously this technique requires a considerable amount of calculations in order to obtain the gain matrix $\mathbf{K}$. Lack of information on the vibration statistics makes it unrealistic to pursue this approach further at this time.

Alignment Time and Accuracy

Without any uncertainty in the measurement of the misalignment angles and with decoupled error channels the governing equations are

$$\dot{\epsilon}_j + k_j\epsilon_j = -(u)\omega_j, \qquad j = N,E,D$$

The time constant is $\tau_j = 1/k_j$. Thus, k_j would be chosen as large as practical in this case and the alignment time could be made very short. A large k_j also gives a small steady-state value of ϵ_j since

$$\epsilon_{j_{ss}} = -(u)\omega_j/k_j$$

However, ϵ_j cannot be obtained without uncertainty in a practical situation. The alignment time therefore mainly depends on the time needed to make a satisfactory estimate $\hat{\epsilon}_j$, and to a lesser extent on the initial value of ϵ_j. The time needed for the estimation on the other hand, depends upon the frequency content of the measurement noise. If only the constant component of the uncertainty is taken into account, the uncertainty in the azimuth error measurement is given by

$$\delta\epsilon_D = (u)\omega_E/\omega_{ie}\cos L + \tan(L)\ (u)f_E/g$$

and the total steady-state azimuth error becomes

$$\epsilon_{D_{ss}} = -(u)\omega_D/k_D + (u)\omega_E/\omega_{ie}\cos L + \tan L\ [(u)f_E/g]$$

The level error can be found in a similar way and the steady-state error due to accelerometer bias and gyro drift are

$$\epsilon_{N_{ss}} = -(u)\omega_N/k_N - (u)f_E/g$$

$$\epsilon_{E_{ss}} = -(u)\omega_E/k_E + (u)f_N/g$$

Conclusions

It has been shown that the analytic method of alignment is mainly useful for obtaining an average transformation between the two frames. Since no linearization has been used, this method is well suited for calculating an initial estimate of the transformation matrix.

The second method, that has been analyzed by linearizing about the true alignment position, is better suited for fine alignment, since compensation for the disturbances is provided in addition to the alignment control.

In both cases, the alignment time is dependent on the time it takes to filter the measurement noise satisfactorily from the signal. Analogous to the physical gyrocompass, the accuracy that can be achieved is determined by the gyro drift and accelerometer bias.

A logical way of combining these two methods is to use the first one to compute an approximate initial alignment matrix that is then refined by the second method.

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